**Recitation 3**

**Problem 1.**

Prove that breaking up an by chocolate bar using only horizontal or vertical breaks requires the same name number of breaks no matter which sequence it is broken up in.

**Lemma 1**: Each increase in size of a chocolate bar must be in terms of or squares.

*Proof.* Imagine we are building a chocolate bar of dimensions *mn*. It can only increase in positive integers, and either *m* or *n* can increase each step. If *m* increases by one, then the bar gains *n* squares. If *n* increases by one, then the bar gains *m* squares. So each step the bar must increase by either *m* or *n* squares.

**Corollary:** Each decrease (break) in size of a chocolate bar must be in terms of or squares.

*Proof.*By Lemma 1, and since each break must be horizontal or vertical along the squares, then a decrease must also reduce the size by *m* or *n* squares.

**Lemma 2**: A chocolate bar of individual squares is broken up in breaks, for all positive integers .

*Proof.* By strong induction. Let be Lemma 2, for which and each increase in is by either or as shown in Lemma 1.

*Base case*. The bar has one square. The only factor of 1 is itself, so . There are no squares to break off, so this is correct.

*Inductive Step*. Let be the inductive hypothesis for all sizes of the chocolate bar up to . From it we will show equals either or in the cases it increases by or respectively.

At size , an bar requires breaks. Then, by Lemma 1, at size there are two possible cases:

Case 1. The size increases by . This row of squares could itself by broken off the bar in 1 break. Since it then consists solely of a row of squares, . So this case adds breaks to the bar. Then

so holds.

Case 2. The size increases by and by the same reasoning this leads to 1 additional break off the bar and more breaks to split it into pieces. Then

showing that also holds for an increase of .

Therefore, no matter in which way the size of the bar increases, the formula for determining the number of breaks the bar requires remains true.

**Theorem**. No matter which sequence the bar is broken up in, the total number of breaks required to reduce it to individual squares remains the same.

*Proof*. By Lemma 2, all sizes of a chocolate bar differing only in terms of or require a number of breaks by the same formula. The formula is not affected by how the bar is broken up. And since the bar can only be decreased in size by or according to the corollary, that formula will always apply to bars reached from any sort of breaking. Therefore, it does not matter which sequence the bar is broken up in.

**Problem 2**.

Each monk enters the Temple of Forever with a bowl of 15 red beads and 12 green beads . Each time the Gong of Time rings, a monk must do one of two things: 1) Exchange (E) 3 red beads for 2 green beads (if at least 3 red). 2) Swap (S) all red beads for green beads and all green beads for red beads.

We will represent each state as a pair for .

is written as

is written as

**Theorem 1**. No one ever leaves the Temple of Forever.

*Proof*. By induction on the number of turns. Let be the proposition that after rings, the values of and will have a difference such that the number of red beads minus the number of green beads is equal to or for some integer .

*Base case*. At the initial state there are 15 red beads and 12 green beads. The difference will be

*Inductive step.* Now we assume holds after number of gong steps. On the th turn equaled or On the th turn, the monks have two options:

Case 1. They do an exchange. So . Substituting in with the difference from the previous turn, we get or . Either way the 5 can be factored to give or , so holds.

Case 2. The monks try a swap. This inverts the difference. or . This could be rewritten as or , so again holds.

Therefore, in all cases implies

By the inductive hypothesis is true for all . Because describes a property that applies to all states of the machine, is an invariant. Then, it can be seen that the target state (5,5) has a difference of 0, and 0 is not possible by or , therefore the target state cannot be reached no monks ever leave the Temple of Forever.

**Theorem 2**. There is a finite number of reachable states in the Temple of Forever machine.

*Proof*. By induction on the number of turns . Let be that after every turn the sum of marbles is less than or equal to the previous turn.

*Base case*. There was no prior sum, so does not apply.

*Inductive step*. On the th turn there are marbles. On the next turn the monks have two choices.

Case 1. The monks do an exchange. so holds.

Case 2. The monks do a swap. so the sum stays the same and holds.

Therefore, in all cases implies By the inductive hypothesis, is true for all . We can then reason that exchanges add a limited number of states, because the starting sum of marbles is finite, and it can’t be applied when . Swaps also add a finite number of states, because a swap can only be applied to a state once without returning to the original state. Therefore there are a finite number of reachable states in the Temple of Forever machine.

**Theorem 3.** It is not possible to visit 108 unique states in the Temple of Forever machine.

*Proof*. Starting with 27 total marbles, a monk can only do an exchange a maximum of 25 times, by the invariant proven in Theorem 2. That gives a start state plus 25 new states giving 26 states. A swap could be applied once to all of these, giving a total of 52 possible states. A swap could not be applied again without repeating a previous state. Therefore it is not possible to visit 108 unique states in the Temple of Forever machine.